Holder's Inequality

Aim

To prove Holder's inequality i.e.

$$(x_1^p + x_2^p + \cdots + x_n^p)^{1/p} \cdot (y_1^q + y_2^q + \cdots + y_n^q)^{1/q} \geqslant x \cdot y$$

where p and q are numbers such that $1/p + 1/q = 1$.

Hint I got:

First try proving

$$x^p/p + y^q/q \geqslant x.y$$

where p and q satisfy $1/p + 1/q = 1$.

Call this the **basic inequality.**

Deal with the hint:

For $p = q = 2$, this is just the simple AM-GM for two quantities. We take motivation from that, and remove pesky fractions first by letting N be an integer such that $N/p, N/q$ are both integers; denote $P_1 = N/p$ and $Q_1 = N/q$. The inequality to be proven then translates to

$$(P_1 x^p + Q_1 y^q)/N \geqslant x.y$$

of course note that $N = P_1 + Q_1$. Now it should be clear that this follows from AM-GM. Take P_1 quantities equal to x^p and Q_1 quantities equal to y^q.

The case of equality: is achieved when $x^p = y^q$.

From the hint to Holder:

Let us take $n = 2$ just to warm up to see what really happens.

The desired inequality is

$$(x_1^p + x_2^p)^{1/p} \cdot (y_1^q + y_2^q)^{1/q} \geqslant x.y$$

We can write down two inequalities (one for x_1 and y_1 and the other for x_2, y_2) as specified by the basic inequality, and add the two inequalities together to get:

$$\frac{x_1^p + x_2^p}{p} + \frac{y_1^q + y_2^q}{q} \geqslant x.y$$

But this is not good enough, the LHS of this inequality is in fact larger than the LHS of Holder; this follows by another application of the *basic inequality*. So we are stumped.

Another Try:

Terry Tao and others suggest that in order to milk the most out of an inequality try to apply it to the scenario where the case of equality can potentially hold. Thus we realise that we would not be able to get from the *basic inequality* to Holder as above, because the case of equality can be widely flouted ($x_1^p + x_2^p$ may be widely different from $y_1^q + y_2^q$). How do we mend matters?

This is where Lagrangian multipliers help. Take a Lagrangian multiplier λ as follows:

$\frac{x_1^p + x_2^p}{p} + \lambda^q \cdot \frac{y_1^q + y_2^q}{q} \geqslant \lambda x.y$

Now we can at least hope to achieve the case of equality, and we set λ accordingly:

$$\lambda^q = \frac{x_1^p + x_2^p}{y_1^q + y_2^q}$$

This expression looks quite formidable, but we are almost there! From the above, the expression on the LHS now reads simply as $(x_1^p + x_2^p)$ while the RHS reads as $\lambda x.y$. So then we have that:

$$\frac{x_1^p + x_2^p}{\lambda} \geqslant x.y$$

and now we simplify this to:

$$\frac{x_1^p + x_2^p}{(x_1^p + x_2^p)^{1/q}} \cdot (y_1^q + y_2^q)^{1/q} \geqslant x.y$$

and noting that $1 - 1/q = 1/p$, we are done:

$$(x_1^p + x_2^p)^{1/p} \cdot (y_1^q + y_2^q)^{1/q} \geqslant x.y$$

The case of n quantities throws up no new surprises, and so we have proven Holder's inequality for ourselves.

Bigger picture

The basic inequality can be interpreted as the Fenchel Young inequality for the norm function x^p/p. The inequality then just says that the Fenchel dual of this function is y^q/q where $1/p + 1/q = 1$. From the Fenchel Young Inequality, we are deriving Holder's inequality.

The bigger question is whether the Fenchel Young inequality is in a sense stronger than Holder's inequality. Holder's inequality is essentially a statement about *dual norms*. Given a norm $\|\cdot\|$ and its dual $\|\cdot\|_*$, Holder's inequality states:

$$\|x\| \cdot \|y\|_* \geqslant x.y$$

Fenchel Young Inequality on the other hand deals with the *Fenchel dual* of functions; given a function f and its Fenchel dual f^*, it holds that

$$f(x) + f^*(y) \geqslant x.y$$

For every norm $\|\cdot\|$ is there a corresponding function f, such that the following chain holds?

$$f(x) + f^*(y) \geqslant \|x\| \cdot \|y\|_* \geqslant x.y$$